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THE NATURE OF RANDOM DEMAND

BUREAU OF SUPPLIES AND ACCOUNTS
NAVY DEPARTMENT

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I. INTRODUCTION

This report was prepared for the Advanced Logistics Research Division, Bureau of Supplies and Accounts, U. S. Navy, in partial fulfillment of Navy Contract Nonr 2904(00).

A major goal of the Navy is to develop scientific inventory decision rules that will reduce supply system costs while maintaining system effectiveness. As an important step in developing these rules, the Navy is devoting much effort to the construction of mathematical models that describe the cost and service effects of inventory policies. The utility of the rules in minimizing system cost depends upon the ability of the mathematical models to approximate real-world behavior.

The essential components of a mathematical model of an inventory system are:1/

1. The direct costs of operating the system, e.g., holding costs, transportation costs, procurement order costs.
2. The imputed costs of departing from system effectiveness criteria, e.g., stockout costs.
3. The administrative costs of managing the system, e.g., data collection, transmittal and processing costs.
4. The constraints imposed on the system, e.g., warehouse capacities, budget limitations.
5. The demand process that the system must satisfy, e.g., periodic replenishment, random failures.

1/ See for example, Transportation-Inventory Trade-offs, Phase B, Decision Models for Supply System Operations, prepared for Bureau of Supplies and Accounts under Navy Contract Nonr 2904(00), by United Research Inc., 1960.

In this report we are concerned with mathematical characterizations of only the demand processes. In the majority of supply situations, the quantity that will be demanded in a given time period is uncertain. Consequently, the behavior of demand with time must be considered to be a random process, describable only in terms of probability distributions. Our primary interest is in the exploration of probability distributions for describing demand that are useful in constructing inventory models.

Ideally, all Navy Supply System items could be described by a single distribution having but one parameter, say average demand per time period. The nature of replacement demand, however, varies widely among items in the system. Some items are consumed in a regular manner that can be predicted with precision, such as food items, and items that are not actually consumed, but may be replaced periodically, or arbitrarily, such as submarine batteries and paint. Other items are consumed in a purely random fashion. This is the case, for example, with certain ship parts, or items that have working lives much longer than useful lives and need replacement due to "accidental" failures. Hatch covers, ship's propellers and shafts, and ladders are items in this category.

It is thus possible to distinguish several classes of items that differ from each other not only in their average demand per time period, but also in their variability of demand. The most commonly used mathematical model which assumes that the quantity demanded will always be one unit and that a demand is as likely in any one short time period as in any other, is not capable of describing the behavior of all these classes.

Consequently, investigators have sought more general mathematical models capable of providing a better description of the observed demand processes. Two of the models that have frequently been suggested are the negative binomial and the stuttering Poisson.

To understand why such distributions have been favored over the infinite number of possible demand models, it is necessary to consider the factors that must influence the investigator in selecting a distribution. It is of primary importance to recognize that any probability distribution selected to describe a demand process will be an approximation. It is an approximation in the sense that there is no experimental method for determining with absolute certainty the fact that the model corresponds to reality. Nevertheless, an approximation is an adequate representation of the process if there is a high probability that the past history of the demand process could have been generated by the model. If the past history of demand is an improbable realization of the process described by the model, then the course of prudence is to reject the model. On the other hand, there may be many probabilistic models that could have generated the observed demand with high probability; in this case, statistical considerations alone cannot serve as a basis for deciding which of these models is an appropriate representation of the demand process.

The use of empirical distributions of demand in inventory system models is seen to be especially unsound in view of the preceding argument. The past history of demand is but a single realization of the demand process, and the accumulation of future demand data may show that it was an extremely improbable realization.

A second important factor in selecting demand distributions is that, for practical reasons, only those approximating distributions that are relatively simple in form can be considered to be satisfactory. There are several reasons why we seek simplicity, in the sense of mathematical tractibility and description of item demand by a limited number of parameters. Mathematically tractible demand distributions are prerequisites to the mathematical analysis of inventory systems. In fact, only the most elementary properties of an inventory system can be determined mathematically if the demand distribution of the system cannot be considered to belong to a surprisingly small class of probability distributions. Furthermore, the development of control rules is made almost impossibly difficult unless this condition is met.

Limitation of the number of parameters of the distribution is required because the limited amount of data available in most inventory situations is sufficient to yield statistically significant estimates of only a few parameters. To the extent that the models contain more parameters than can be estimated with assurance, the investigator would be better advised to use a simpler demand model until more information on the demand process is accumulated. Since in most demand situations, only the mean and variance can be meaningfully measured or predicted, only those models whose parameters are determinable from these quantities should be employed.

Another important factor in selecting a demand distribution is associated with simplicity but not directly related to it. This factor is the physical interpretability of the probability distributions used to describe demand. A major distinction between the statistician or mathematician and the operations analyst lies in their regard for the physical interpretability of the results. The mathematician is primarily concerned with the "fit" of his theoretical distribution to the emprical distribution. If the fit is good in a statistical sense, then the mathematician can usually decide that he has obtained a satisfactory description of the process. The operations analyst, on the other hand, requires not only mathematical significance, but also physical reasonableness of the results. The operations analyst first observes the physical process of demand and then hypothesizes a microscopic model of demand. He determines the over-all demand distribution that would be generated by the microscopic model, and, at this point, establishes the validity of his characterization of the process by comparing the statistical properties of his model with those of the real world.

The distinction between the two approaches is not trivial. The requirement of physical reasonableness helps to assure the operations analyst that he has not ignored some important physical aspects of the problem in formulating the mathematical model. Just as one would be wary of driving over bridges, or of flying in airplanes designed entirely by mathematicians, so should he be cautious in accepting the results of an inventory model based on mathematical considerations alone.

II. SUMMARY

The purpose of this report is to examine the probability distributions that are pertinent to the description of Navy replacement demand processes. Demand patterns observed in the Navy Supply System indicate that the actual demand process is complicated. Simple Poisson models for demand appear to be useful only in special situations, such as for certain classes of items that have very low demand rates. The remaining situations indicate that there is more "clumping" of demand or regularity of time between demands than is predicted by the Poisson model.

Consequently, various investigators have proposed other models that allow for a range of variance-to-mean ratios and other statistical parameters of demand to capture the basic nature of these more complex patterns. In particular such distributions as the negative binomial, stuttering Poisson, geometric Poisson and the Erlang appear frequently in the literature of logistics systems, and are discussed in this report. Scrutiny of these models has revealed that the class of models that has been considered for describing the demand process is not so large as it might appear because some of the models are special cases of others. In particular the following properties of commonly used demand distributions do not seem to be generally appreciated by inventory systems analysts:

1. The assumption that demand in any time period is statistically independent of demand in any other non-overlapping time period implies that the demand must be described by a member of the class of distributions known as compound Poisson.1/

1/ Although some investigators use the term compound Poisson to refer to the stuttering Poisson distribution, this usage is misleading since many other distributions are also compound Poisson distributions.

2. Although the stuttering Poisson and the geometric Poisson distributions are apparently considered to be essentially different by many investigators, there is no difference when the initial conditions of the demand processes are appropriately selected.

3. The negative binomial distribution, which is frequently advocated as a demand distribution, is interpretable as a compound Poisson distribution in most situations. Furthermore, there are several microscopic demand models that can lead to overall demand distributions of the negative binomial family. In accordance with the earlier discussion of physical reasonableness, the appropriateness of the negative binomial in describing a demand process can be evaluated by determining whether one of these microscopic models is a reasonable source of demand.

In the remainder of this report, a systematic analysis of demand distributions will be presented. This analysis will establish not only the results indicated, but will also consider demand distributions from a more general point of view.

III. COMPOUND DISTRIBUTIONS

Introduction

The largest class of probability distributions useful for the description of demand appears to be the class of compound distributions. These distributions are capable of describing fairly general demand processes often without a sacrifice of analytic tractibility. Simplicity in mathematical analysis is most likely to exist where the demand process is of the generalized Poisson family that is usually assumed explicitly or implicitly in the literature of inventory control. This chapter develops the theory of compound distributions. In Chapter IV the theory of recurrent events is developed as a foundation for later chapters. Chapter V presents the theory of compound Poisson processes as an important family of compound distributions. A second family of distributions that may be useful in the description of demand, the Erlang process, is suggested in Chapter VI.

Development of the Compound Distribution and its z-Transform

The process by which compound distributions are generated is described as follows. Consider two independent random variables n and k with density functions $f(n)$ and $g(k)$. The functions $f(n)$ and $g(k)$ will be called the compounding and sampling distributions respectively. The function $g(k)$ can have non-zero values only for non-negative integers, k. The function $f(n)$ could have as its domain any real number, but we shall restrict its domain to the integers for our present discussion.[1/] In accordance with the integral domains of the compounding and sampling distributions it is possible for us to think of these distributions as probability mass functions rather than probability

1/ The extension of the argument to the case where n can be a continuous variable is developed in the appendix to this report.

density functions, however we shall use the latter form.

Let m be the sum of k independent random samples from $f(n)$, where k is a random variable from $g(k)$. The problem is to find $h(m)$, the density function of m. The function $h(m)$ is called the compound distribution generated by the functions $f(n)$ and $g(k)$. Let $h(m|k)$ be the probability that the sum of k samples from $f(n)$ takes on the value m. Then we may write:

$$h(m) = \sum_{k=0}^{\infty} g(k)\, h(m|k) \quad . \tag{3.1}$$

Since m is the sum of k independent samples of $f(n)$ the probability $h(m|k)$ is simply the k-fold convolution of $f(n)$, denoted by $f^{k*}(m)$. The quantity $f^{k*}(m)$ is defined by

$$f^{k*}(m) = \sum_{j=-\infty}^{\infty} f(j)\, f^{(k-1)*}(m-j) \qquad k=1,2,\ldots \quad m=0,\pm 1,\pm 2,\ldots$$

$$f^{0*}(m) = \begin{cases} 1 & \text{for } m=0 \\ 0 & \text{elsewhere} \end{cases} \tag{3.2}$$

Now Equation (3.1) may be written:

$$h(m) = \sum_{k=0}^{\infty} g(k)\, f^{k*}(m) \tag{3.3}$$

Let $H(z)$, $G(z)$, and $F(z)$ be the z-transforms of the density functions $h(m)$, $g(k)$, and $f(n)$, respectively, defined by

$$H(z) = \sum_{m=-\infty}^{\infty} z^m h(m) \quad ,$$

$$G(z) = \sum_{k=0}^{\infty} z^k g(k) \quad , \tag{3.4}$$

$$F(z) = \sum_{n=-\infty}^{\infty} z^n f(n) \quad .$$

If we multiply both sides of Equation 3.3 by z^m and sum over all m, we obtain

$$\sum_{m=-\infty}^{\infty} h(m)\, z^m = \sum_{k=0}^{\infty} g(k) \sum_{m=-\infty}^{\infty} z^m f^{k*}(m) \quad . \tag{3.5}$$

Using the definitions given in Equations 3.4 and the fact that the z-transform of the k-fold convolution of a probability distribution is equal to its z-transform raised to the k^{th} power we may write Equation 3.5 as

$$H(z) = \sum_{k=0}^{\infty} g(k)\, [F(z)]^k \quad . \tag{3.6}$$

The definition of $G(z)$ allows us to write Equation 3.6 in the form

$$H(z) = G\,[F(z)] \quad . \tag{3.7}$$

Equation 3.7 is our basic result. It shows that the z-transform of the compound distribution may be obtained by taking the z-transform of the distribution that determines the number of samples (the sampling distribution) and replacing every z by the z-transform of the distribution from which the samples are being taken (the compounding distribution).

The demand in a given time period is described by the compound distribution, $h(m)$, defined by Equation 3.3, or equivalently in transform terms by Equation 3.7, if the number of customers that arrive in a time period, k, is described by the density function $g(k)$ and the number of units, n, that each customer demands is independent of k and is governed by the density function $f(n)$. Although n, the quantity ordered by a customer, will usually be positive, it can take on negative values if customers are able to return supplies previously received but not used.

Moments of the Compound Distribution

The moments of the compound distribution may be related to the moments of the sampling distribution and the moments of the compounding distribution. To obtain this relation, let us recall that if we have a distribution $f(n)$ with z-transform $F(z)$, the mean, $\bar{n}$, second moment, $\overline{n^2}$, and variance, σ_n^2, of $f(n)$ are related to the derivatives of the z-transform $F(z)$ at the point $z=1$ by the equations:

$$\bar{n} = \left.\frac{dF(z)}{dz}\right|_{z=1} = F'(1) \quad ,$$

$$\overline{n^2} = \left.\frac{d^2F(z)}{dz^2}\right|_{z=1} + \left.\frac{dF(z)}{dz}\right|_{z=1} = F''(1)+F'(1) \tag{3.8}$$

and

$$\sigma_n^2 = \overline{n^2} - \bar{n}^2 = F''(1) + F'(1) - [F'(1)]^2$$

The mean, $\bar{m}$, and variance, σ_m^2, of the compound distribution, $h(m)$, may then be obtained as follows:

$$H(z) = G\,[F(z)]$$

$$H'(z) = G'[F(z)]\; F'(z)$$

$$H'(1) = G'[F(1)]\; F'(1) \quad . \tag{3.9}$$

However,

$$F(1) = 1 \quad , \tag{3.10}$$

so that we obtain

$$H'(1) = G'(1)\, F'(1)$$

or

$$\bar{m} = \bar{k}\, \bar{n} \quad . \tag{3.11}$$

Equation 3.11 shows that the mean of the compound distribution is equal to the product of the means of the sampling distribution and the compounding distribution.

For the variance, σ_m^2, we first compute

$$H''(z) = \left\{ G'[F(z)]\;\; F'(z) \right\}'$$

$$= G''[F(z)]\;\; F'(z) \cdot F'(z) + G'[F(z)]\;\; F''(z)$$

$$H''(1) = G''(1)\, [F'(1)]^2 + G'(1)\, F''(1)$$

$$= G''(1)\bar{n}^2 + \bar{k}\; F''(1) \quad . \tag{3.12}$$

The variance is given by:

$$
\begin{aligned}
\sigma_m^2 &= H''(1) + H'(1) - [H'(1)]^2 \\
&= G''(1)\,\bar{n}^2 + \bar{k}\,F''(1) + \bar{k}\,\bar{n} - \bar{k}^2\,\bar{n}^2 \\
&= [\sigma_k^2 - \bar{k} + \bar{k}^2]\ \bar{n}^2 + \bar{k}\,[\sigma_n^2 - \bar{n} + \bar{n}^2] + \bar{k}\,\bar{n} - \bar{k}^2\,\bar{n}^2 \\
&= \bar{n}^2\,\sigma_k^2 + \bar{k}\,\sigma_n^2 \qquad (3.13)
\end{aligned}
$$

We see from Equation 3.13 that the variance of the compound distribution is equal to the variance of the sampling distribution multiplied by the square of the mean of the compounding distribution plus the variance of the compounding distribution multiplied by the mean of the sampling distribution.

IV. RECURRENT PROCESSES

Introduction

The previous chapter discussed the theory of compound distributions because they are important models for the number of purchases in a given time period. In this chapter we shall investigate, in detail, the time behavior of demand processes. We are interested in counting the number of events that occur in a given time period when the distribution of times between arrivals does not vary with the passage of time. The results of the chapter show that the distribution of the number of events in a given time period depends on the definition of the time interval over which the events occur. The basic theory underlying the type of demand model we are discussing is the theory of recurrent events.

If we define the arrival of a customer to be an event, then for stationary processes where customer arrival times are independent we can define the density function of the time between successive events, or the interevent time, τ, by $a(\tau)$. By repeated sampling of the density function $a(\tau)$ we can construct a realization of the customer arrival process. An important statistic of this process is the number of events in a given time interval. To define this statistic precisely it is necessary to specify how the time interval is placed with respect to the occurrence of events. One way to define this interval is to say that it begins immediately after the occurrence of an event and lasts for a time interval t.

Counting from an Event

Let us define $p(n, t)$ to be the probability of n events in such a time interval t. The number of events that occur in a time interval t following an event will be $n \geq 1$ if the first event occurs at a time τ $(\tau < t)$ after the beginning of the interval, and $n-1$ events occur in the remaining time period, $t-\tau$, which, of course, is initiated by an event. Since the occurrence of the first arrival at a time τ represents a mutually exclusive set of events for different values of $\tau<t$, and since the occurrence of the first arrival at time τ is independent of the occurrence of the remaining arrivals in the interval $t-\tau$, we may write

$$p(n,t) = \int_0^t a(\tau)\, p(n-1,\, t-\tau)\, d\tau \quad , \qquad n \geq 1 \ . \tag{4.1}$$

The probability that there will be n events in a time interval t beginning with an event is thus the convolution of the interevent time density function, $a(\tau)$, with $p(n-1,t)$.

Let us denote by $P(n,s)$ and $A(s)$ the Laplace transforms of $p(n,t)$ and $a(\tau)$, defined by

$$P(n,s) = \int_0^\infty p(n,t)\, e^{-st}\, dt = L\,[p(n,t)] \quad ,$$

and (4.2)

$$A(s) = \int_0^\infty a(\tau)\, e^{-s\tau}\, d\tau = L\,[a(\tau)] \quad .$$

In the transform domain, the convolution of two functions becomes the product of their transforms, and hence Equation 4.1 may be written as the difference equation

$$P(n,s) = A(s)\, P(n-1,s) \quad , \qquad n \geq 1 \ . \tag{4.3}$$

The solution to this difference equation is

$$P(n,s) = P(0,s)\, [A(s)]^n \quad , \qquad n \geq 1 \ . \tag{4.4}$$

It now remains to find $P(0,s)$, the transform of $p(0,t)$. No arrivals will occur in a time interval t beginning with an event if the time until the next event τ is greater than t. Hence, we may write

$$p(0,t) = \int_t^\infty a(\tau)\, d\tau$$

$$= 1 - \int_0^t a(\tau)\, d\tau \quad . \tag{4.5}$$

Taking the Laplace transform of Equation 4.5 we obtain

$$P(0,s) = \frac{1}{s} - \frac{A(s)}{s} \quad . \tag{4.6}$$

Thus, we can write Equation 4.4 as

$$P(n,s) = \frac{1}{s}\,[1 - A(s)]\,[A(s)]^n \qquad n \geq 0 \tag{4.7}$$

Equation 4.7 yields the Laplace transform of p(n,t) for an arbitrary interevent time distribution, $a(\tau)$.

Counting from an Arbitrary Time

Although it is sometimes important to count events in a time period beginning with an event, it is more often the case that we wish to count events beginning with an initial time selected in a more general way. Let us suppose that counting is started at an arbitrary instant such that the time to the next event, τ_1, has a density function $h(\tau_1)$. This density function $h(\tau_1)$ is generally different from the interevent time density function, $a(\tau)$. Let $p_A(n,t)$ be the probability that n events will occur in a time interval t that begins at such an instant. Then by reasoning similar to that used in obtaining Equation 4.1 we obtain

$$p_A(n,t) = \int_0^t h(\tau_1)\, p(n-1,\, t-\tau_1)\, d\tau_1 \quad , \qquad n \geq 1 \quad . \tag{4.8}$$

Equation 4.8 shows that the probability of n events in a time interval t starting at an arbitrary point in time is given by the convolution of the density function for the time until the first event, $h(\tau_1)$, with the probability that n-1 events will occur in a time t starting after an event. We define $P_A(n,s)$ and $H(s)$ to be the Laplace transforms of $p_A(n,t)$ and $h(\tau_1)$. Then Equation 4.8 may be written in the transform domain as

$$P_A(n,s) = H(s)\ P(n-1,s) \quad , \qquad n \geq 1 \ . \tag{4.9}$$

If we substitute the result of Equation 4.7 into Equation 4.9 we obtain

$$P_A(n,s) = \frac{1}{s}\ H(s)\ [1-A(s)]\ [A(s)]^{n-1} \qquad n \geq 1 \tag{4.10}$$

Equation 4.10 gives the transform of the probability $p_A(n,t)$ for $n \geq 1$. It remains to find the transform of $p_A(0,t)$. By reasoning similar to that used for Equation 4.5 we obtain

$$p_A(0,t) = 1 - \int_0^t h(\tau_1)\ d\tau_1 \quad . \tag{4.11}$$

The Laplace transform of Equation 4.11 is

$$P_A(0,s) = \frac{1}{s}\ [1-H(s)] \quad . \tag{4.12}$$

If $H(s) = A(s)$ the results of Equations 4.10 and 4.12 agree with Equation 4.7.

Counting from a Time Selected at Random

Perhaps the most important case of counting for recurrent processes arises when the initial time is selected independently of the event process; a situation most often called starting "at random" in time. This way of starting the process will imply a particular form of $h(\tau_1)$, the density function for the time until the next arrival, and so is a special case of the arbitrary starting process described above.

Let us define $h_R(\tau_1)$ to be the density function for the time until the first event from a time selected at random, and let $H_R(s)$ be its Laplace transform. Furthermore, let $p_R(n,t)$ be the probability of n events in a time t starting at random, and let $P_R(n,s)$ be its Laplace transform. From these definitions and Equations 4.10 and 4.12

$$P_R(n,s) = \begin{cases} \frac{1}{s} H_R(s) \, [1 - A(s)] \, [A(s)]^{n-1} , & n \geq 1 \\ \frac{1}{s} [1 - H_R(s)] , & n = 0 \end{cases} \tag{4.13}$$

To find $h_R(\tau_1)$ we first recall that the average time between events, $\bar{\tau}$, is given by

$$\bar{\tau} = - \frac{d}{ds} A(s) \Big|_{s=0} , \tag{4.14}$$

and therefore that the average event rate, λ, is

$$\lambda = \frac{1}{\bar{\tau}} = - \frac{1}{A'(0)} . \tag{4.15}$$

If the starting point is selected at random, the next event will occur at a time between τ_1 and $\tau_1 + d\tau_1$ if an event occurs in the interval $d\tau_1$ and if the time between this event and the preceding event is greater than τ_1. Since the probability of an event in any short time interval is $\lambda d\tau_1$, the previous statement in probability terms is

$$\begin{aligned} h_R(\tau_1) \, d\tau_1 &= \lambda d\tau_1 \left[\text{Prob interevent time} > \tau_1 \right] \\ &= \lambda d\tau_1 \int_{\tau_1}^{\infty} a(\tau) \, d\tau , \end{aligned}$$

or

$$h_R(\tau_1) = \lambda \left[1 - \int_0^{\tau_1} a(\tau) \, d\tau \right] . \tag{4.16}$$

Equation 4.16 is an explicit expression for $h_R(\tau_1)$ in terms of the interevent time density function $a(\tau)$. The Laplace transformation of Equation 4.16 is

$$H_R(s) = \frac{\lambda}{s} [1 - A(s)] \quad . \tag{4.17}$$

Finally, substitution of Equation 4.17 into Equation 4.13 yields

$$P_R(n, s) = \begin{cases} \frac{\lambda}{s^2} [1 - A(s)]^2 [A(s)]^{n-1} , & n \geq 1 \\ \frac{1}{s^2} [s - \lambda + \lambda A(s)] , & n = 0 \end{cases} \quad . \tag{4.18}$$

Bivariate Transform

It is often useful to transform with respect to the discrete variable, n, in our probability expressions as well as with respect to the continuous variable t, i.e., it is useful for calculating the moments of the distribution of the number of events in a time interval t. Let us define the bivariate transform $P^b(z, s)$ of $p(n, t)$ by

$$P^b(z, s) = \sum_{n=0}^{\infty} z^n \int_0^{\infty} p(n, t) \, e^{-st} \, dt$$

$$= \sum_{n=0}^{\infty} z^n \, P(n, s) \quad . \tag{4.19}$$

For $P(n, s)$ given by Equation 4.7 we have

$$P^b(z,s) = \sum_{n=0}^{\infty} z^n \frac{1}{s} [\, 1-A(s)\,] \, [\, A(s)\,]^n$$

$$= \frac{1}{s} [\, 1-A(s)\,] \sum_{n=0}^{\infty} [\, zA(s)\,]^n$$

$$= \frac{1-A(s)}{s[\, 1-zA(s)\,]} \tag{4.20}$$

Similarly, we shall define $P_A^b(z,s)$ and $P_R^b(z,s)$ to be the bivariate transforms of $p_A(n,t)$ and $p_R(n,t)$. From Equations 4.10 and 4.12 for $p_A(n,s)$ we have

$$P_A^b(z,s) = \frac{1}{s} [\, 1-H(s)\,] + \sum_{n=1}^{\infty} z^n \frac{1}{s} H(s) [\, 1-A(s)\,] \, [\, A(s)\,]^{n-1}$$

$$= \frac{1-H(s)}{s} + \frac{H(s)}{s} [\, 1-A(s)\,] \; z \sum_{j=0}^{\infty} [\, zA(s)\,]^j$$

$$= \frac{1-H(s)}{s} + \frac{zH(s)[\, 1-A(s)\,]}{s[\, 1-zA(s)\,]}$$

$$= \frac{1-H(s)-zA(s)+zH(s)}{s[\, 1-zA(s)\,]} \tag{4.21}$$

For counting from a random time we can substitute Equation 4.17 for H(s) in Equation 4.21

$$P_R^b(z,s) = \frac{1-\frac{\lambda}{s}[1-A(s)] - zA(s) + z\frac{\lambda}{s}[1-A(s)]}{s[1-zA(s)]}$$

$$= \frac{s-\lambda + \lambda A(s) - zsA(s) + z\lambda - z\lambda A(s)}{s^2[1-zA(s)]}$$

$$= \frac{(s-\lambda + z\lambda) + A(s)\,[\lambda - zs - z\lambda]}{s^2[1-zA(s)]} \tag{4.22}$$

Use of the Bivariate Transform to Calculate Moments

The bivariate transform $P^b(z,s)$ is an important quantity because it may be used to determine all the moments of the distribution of the number of events in time t. In particular, the Laplace transform of $\bar{n}(t)$, the expected number of events in time t, is found by evaluating the derivative of $P^b(z,s)$ with respect to z at the point z=1. This is shown as follows

$$P^b(z,s) = \sum_{n=0}^{\infty} z^n \int_0^{\infty} p(n,t)e^{-st}dt$$

$$\frac{\partial}{\partial z} P^b(z,s) = \sum_{n=0}^{\infty} nz^{n-1} \int_0^{\infty} p(n,t)e^{-st}dt$$

$$\left.\frac{\partial}{\partial z} P^b(z,s)\right|_{z=1} = \int_0^{\infty} e^{-st} \sum_{n=0}^{\infty} np(n,t)dt$$

$$= \int_0^{\infty} \bar{n}(t)\, e^{-st}\, dt$$

$$= L[\bar{n}(t)] \tag{4.23}$$

For a recurrent event process starting after an event we may calculate the Laplace transform of the expected number of events from Equation 4.20 as

$$\frac{\partial}{\partial z} P^b(z,s) = \frac{A(s)[1-A(s)]}{s[1-zA(s)]^2} \quad ,$$

and

$$L[\bar{n}(t)] = \frac{\partial}{\partial z} P^b(z,s)\Big|_{z=1} = \frac{A(s)}{s[1-A(s)]} \quad . \tag{4.24}$$

Similarly, for a process starting at an arbitrary time we may use Equation 4.21 to obtain the expected number of events in a time t, $\bar{n}_A(t)$, as follows

$$\frac{\partial}{\partial z} P_A^b(z,s) = \frac{H(s)[1-A(s)]}{s[1-zA(s)]^2} \quad ,$$

and

$$L[\bar{n}_A(t)] = \frac{\partial}{\partial z} P_A^b(z,s)\Big|_{z=1} = \frac{H(s)}{s[1-A(s)]} \quad . \tag{4.25}$$

Finally, to calculate the Laplace transform of the expected number of events in a time t for a process starting at random, $\bar{n}_R(t)$, we substitute $H_R(s)$ from Equation 4.17 into Equation 4.25 to obtain

$$L[\bar{n}_R(t)] = \frac{\lambda}{s^2} \quad . \tag{4.26}$$

If we take the inverse transform of Equation 4.26 we find that

$$\bar{n}_R(t) = \lambda t \quad . \tag{4.27}$$

Equation 4.27 is a particularly important result because it shows that the expected number of events in a time t starting at random is equal to the product of the average event rate, λ, and the time interval t and thus depends only upon the mean of the interevent time distribution.

This chapter has summarized the theory relevant to the counting of recurrent events with particular emphasis on the importance of the starting point of the period. This theory will be utilized in discussing the compound distributions used to describe demand.

V. COMPOUND POISSON

Introduction

The compound Poisson distribution is a family of compound distributions of special importance in inventory control theory. In this chapter we shall develop the properties of the compound Poisson family and show how such commonly used distributions as the stuttering Poisson and negative binomial may be interpreted as compound Poisson distributions. Finally, the point is made that the assumption of independent demand in independent intervals, common in inventory analyses, implies a demand distribution of the compound Poisson type.

The Regular Poisson Process

The Poisson process may be described as a recurrent event process where the interarrival time distribution, $a(\tau)$, is exponential in form,

$$a(\tau) = \lambda e^{-\lambda\tau} \quad . \tag{5.1}$$

The Laplace transform of this distribution is

$$A(s) = \frac{\lambda}{s+\lambda} \quad . \tag{5.2}$$

We are interested in the number of customers that will arrive in a time t. Chapter IV shows that this probability generally depends on the placement of the time interval. In particular, Equation 4.7 gives the Laplace transform of the probability that n customers will arrive in a time t following a customers' arrival, $p(n,t)$. Since we are dealing with a Poisson process we substitute Equation 5.2 into Equation 4.7 and write

$$P(n,s) = \frac{1}{s}\left(1 - \frac{\lambda}{s+\lambda}\right)\left(\frac{\lambda}{s+\lambda}\right)^n$$

$$= \frac{\lambda^n}{(s+\lambda)^{n+1}} \quad , \qquad n \geq 0 \quad . \tag{5.3}$$

Finally, inverse transformation yields

$$p(n,t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \quad . \tag{5.4}$$

Equation 5.4 is an expression for the probability that n customers arrive in a time t following an arrival of a customer. If counting is begun at random then Equation 5.2 must be substituted into Equation 4.18 to yield $P_R(n,s)$, the transform of the probability that n customers will arrive in a time t following a point at random; the result is

$$P_R(n,s) = \begin{cases} \dfrac{\lambda}{s^2}\left(1 - \dfrac{\lambda}{s+\lambda}\right)^2 \left(\dfrac{\lambda}{s+\lambda}\right)^{n-1} , & n \geq 1 \ . \\ \dfrac{1}{s^2}\left(s - \lambda + \dfrac{\lambda^2}{s+\lambda}\right) , & n = 0 \ , \end{cases}$$

or

$$P_R(n,s) = \frac{\lambda^n}{(s+\lambda)^{n+1}} \quad . \qquad n \geq 0 \ . \tag{5.5}$$

Note that Equation 5.5 is the same as Equation 5.3 and therefore $p(n,t) = p_R(n,t)$, and we obtain the same probabilistic behavior of the process for both event-determined starting times and random starting times.

The bivariate transform of $p(n,t) = p_R(n,t)$ is given by

$$\begin{aligned} P^b(z,s) = P_R^b(z,s) &= \sum_{n=0}^{\infty} z^n \frac{\lambda^n}{(s+\lambda)^{n+1}} \\ &= \frac{1}{s+\lambda} \sum_{n=0}^{\infty} \left(\frac{z\lambda}{s+\lambda}\right)^n \\ &= \left(\frac{1}{s+\lambda}\right)\left(\frac{1}{1-\dfrac{z\lambda}{s+\lambda}}\right) \\ &= \frac{1}{s+\lambda - z\lambda} \quad . \end{aligned} \tag{5.6}$$

Definition of the Compound Poisson Process

The compound Poisson distribution arises in a demand situation when customers arrive at random according to a Poisson process and then purchase a number of units, n, governed by a discrete density function, $f(n)$. The probability that k customers will arrive in a time t if the average customer arrival rate is λ is obtained from Equation 5.4 as

$$g(k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad . \tag{5.7}$$

The number of purchases, m, that will be made in a time t is then governed by the compound Poisson density function determined by the Poisson sampling function, $g(k)$, and the arbitrary compounding distribution, $f(n)$.

According to Equation 3.7 the z-transform of the compound distribution, $H(z)$, is related to the z-transform of the sampling distribution, $G(z)$, and the z-transform of the compounding distribution, $F(z)$, by

$$H(z) = G[F(z)] \quad . \tag{5.8}$$

The z-transform of the Poisson sampling distribution is obtained using Equation 5.7 as

$$G(z) = \sum_{k=0}^{\infty} g(k) z^k = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t z)^k}{k!} = e^{-\lambda t} \cdot e^{\lambda t z}$$

or

$$G(z) = e^{-\lambda t(1-z)} \quad . \tag{5.9}$$

Therefore, for an arbitrary compounding distribution $f(n)$ with z-transform $F(z)$, the z-transform of the compound Poisson distribution of purchases, $H(z,t)$, is given by

$$H(z,t) = e^{-\lambda t[1 - F(z)]} \quad . \tag{5.10}$$

The Regenerative Property of the Compound Poisson

Many interesting properties of the compound Poisson distribution as a demand model may be derived from Equation 5.10. Suppose, for example, that we have one compound Poisson demand process with customer arrival rate λ, and a compounding distribution of individual customer purchases, $f_1(n)$, whose z-transform is given by $F_1(z)$. Suppose also that we have a second compound Poisson demand process with corresponding properties λ_2, $f_2(n)$, and $F_2(z)$. If the processes are independent, then the z-transform of the probability density function of the total purchases in time t is given by

$$H(z) = H_1(z)\,H_2(z) = e^{-\lambda_1 t[1-F_1(z)]}\; e^{-\lambda_2 t[1-F_2(z)]} \quad . \tag{5.11}$$

Let us investigate under what conditions Equation 5.11 also represents a member of the compound Poisson family. We may write Equation 5.11 in the following way

$$H(z) = \exp \left\{ -(\lambda_1+\lambda_2)t\,\left[1-\frac{\lambda_1}{\lambda_1+\lambda_2}\,F_1(z)-\frac{\lambda_2}{\lambda_1\lambda_2}\,F_2(z)\right]\right\} \tag{5.12}$$

or

$$H(z) = \exp \left\{ -(\lambda_1+\lambda_2)t[1-F(z)]\right\} \quad . \tag{5.13}$$

We see that Equation 5.13 represents a compound Poisson process with a customer arrival rate equal to the sum of the rates of the two original processes and with a compounding distribution whose z-transform, $F(z)$, is the transform of the sum of the two original compounding distributions, weighted by their respective arrival rates; that is,

$$F(z) = \frac{\lambda_1}{\lambda_1+\lambda_2}\,F_1(z) + \frac{\lambda_2}{\lambda_1+\lambda_2}\,F_2(z) \quad . \tag{5.14}$$

Therefore we have shown that the sums of compound Poisson processes are compound Poisson processes.

A demand model for which the result expressed by Equation 5.13 is relevant is the model for the physical situation where two groups of customers with different arrival rates and purchasing characteristics purchase the same item. For example, destroyers and aircraft carriers will have quite different arrival and purchase characteristics for certain types of vacuum tubes.

Moments of the Compound Poisson Distribution

The moments of the compound Poisson distribution are easily obtained from the results of Chapter III. The mean and variance of the number of customer arrivals in a time t for the Poisson sampling distribution of Equation 5.7 are

$$\bar{k} = \sigma_k^2 = \lambda t \quad . \tag{5.15}$$

The mean number of purchases in time t, $\bar{m}$, is obtained from Equation 3.11 as

$$\bar{m} = \lambda t \bar{n} \quad , \tag{5.16}$$

while the variance is given by Equation 3.13 as

$$\begin{aligned} \sigma_m^2 &= \bar{n}^2 \lambda t + \lambda t \sigma_n^2 \\ &= \lambda t \, (\bar{n}^2 + \sigma_n^2) \\ &= \lambda t \, \overline{n^2} \end{aligned} \tag{5.17}$$

We thus see that the mean and the variance of the compound Poisson distribution are proportional to only the first and second moments respectively of the compounding distributions.

A parameter that is often used to describe demand distributions is the variance-to-mean ratio, R , defined by

$$R = \frac{\sigma_m^2}{\bar{m}} \quad . \tag{5.18}$$

If we use Equations 5.16 and 5.17 we see that for the compound Poisson

$$R = \frac{\overline{n^2}}{\bar{n}} \quad . \tag{5.19}$$

We shall now show that the variance-to-mean ratio for the compound Poisson must be greater than or equal to one, if $\bar{n}$ is positive. The first and second moments of the compounding distribution are defined by

$$\bar{n} = \sum_{n=-\infty}^{\infty} n\, f(n) \quad , \tag{5.20}$$

and

$$\overline{n^2} = \sum_{n=-\infty}^{\infty} n^2 f(n) \quad . \tag{5.21}$$

Consider the difference $\overline{n^2} - \bar{n}$. We may write

$$\overline{n^2} - \bar{n} = \sum_{n=-\infty}^{\infty} n^2 f(n) - \sum_{n=-\infty}^{\infty} n f(n)$$

$$= \sum_{n=-\infty}^{\infty} (n^2 - n) f(n)$$

$$= \sum_{n=-\infty}^{\infty} n(n-1) f(n) \tag{5.22}$$

Since the product $n(n-1)$ is always greater than or equal to zero for any integral value of n and since $f(n)$ can never be negative it follows that

$$\overline{n^2} - \bar{n} \geq 0$$

$$\overline{n^2} \geq \bar{n} \quad ,$$

and finally, since $\bar{n}$ is positive by assumption, we find from Equation 5.19

$$R \geq 1 \quad . \tag{5.23}$$

This completes our discussion of the properties of the compound Poisson distribution. The sections following describe certain special members of the compound Poisson family that have been suggested for demand models.

The Geometric Poisson and the Stuttering Poisson Distributions

Introduction

Two distributions that are often used for describing demand in inventory systems are the "stuttering" Poisson and the geometric Poisson distributions. It is of interest to examine the exact interrelationship between these distributions in order to determine how results of analyses based on one of the distributions may be modified to be applicable to systems analysed using the other.

The stuttering Poisson distribution refers to an arrival process where interarrival times are exponentially distributed except that there is a finite probability of a zero interarrival time represented by an impulse at the origin of the interarrival time density function. If we consider each arrival as a single demand on the system, then the impulse in the interarrival density function will cause a "clumping" of arrivals and hence a clumping of demand that would not be present if an interarrival time of zero had zero probability. (The simple Poisson process is this special case of the stuttering Poisson.) As a result the stuttering Poisson is sometimes a useful model for systems where multiple orders can occur.

The distribution allows clumping of orders in a conceptually different way. For this distribution the interarrival time density function is a pure exponential, but each arrival when it occurs generates a demand by sampling from a geometric distribution. The question arises as to how the clumping of demand caused by this distribution differs, if at all, from that caused by the stuttering Poisson.

Analysis

To examine this question let us examine each distribution in more detail, beginning with the stuttering Poisson distribution. For the stuttering Poisson, the interarrival time, τ, is zero with probability p and is selected from a density function $\mu e^{-\mu\tau}$ with probability 1-p, so that $a(\tau)$, its density function, is given by

$$a(\tau) = p\delta(\tau) + (1-p)\mu e^{-\mu\tau} \quad , \tag{5.24}$$

where $\delta(\tau)$ represents a unit impulse at $\tau=0$. The Laplace transform, $A(s)$, of this density function is then

$$A(s) = p + \frac{(1-p)\mu}{s+\mu} \quad . \tag{5.25}$$

The mean arrival rate is given by Equation 4.15 as

$$\lambda = -\frac{1}{A'(0)} \quad ,$$

and hence

$$\lambda = \frac{\mu}{1-p} \quad .$$

The probability that n customers will arrive in a time t beginning with an arrival has a bivariate transform, $P^b(z,s)$, given by Equation 4.20 using the $A(s)$ from Equation 5.25 above.

$$P^b(z,s) = \frac{1-p}{s+\mu - szp - \mu z} \quad . \tag{5.26}$$

If p is chosen to be zero then the stuttering Poisson distribution becomes the regular Poisson distribution with arrival rate μ, for which $P(z,s) = 1/(s+\mu-\mu z)$, as indicated by Equation 5.6. Equation 5.26 is thus verified for this particular case.

The Laplace transform of the expected number of demands in a time t, $\bar{n}(t)$, is obtained by using Equation 4.24.

$$L[\bar{n}(t)] = \frac{\partial}{\partial z} P^b(z,s) \Big|_{z=1} = \frac{A(s)}{s[1-A(s)]} = \frac{\mu/(1-p)}{s^2} + \frac{p/(1-p)}{s} \tag{5.27}$$

and inverse transformation yields

$$\bar{n}(t) = \frac{\mu}{1-p} t + \frac{p}{1-p} \quad . \tag{5.28}$$

The expected number of demands in time t therefore has a component that increases linearly with time at a rate $\mu/(1-p)$ which is, as we found earlier, λ. It also has a constant component, $p/(1-p)$, so that the expected number of purchases in time zero is equal to $p/(1-p)$. This is a disquieting feature of the stuttering Poisson, but let us proceed to our examination of the geometric Poisson before considering this fact further. If the geometric Poisson also has a bivariate transform $P(z,s)$ given by Equation 5.26, then it will be equivalent to the stuttering Poisson as a demand distribution. If it does not, then we shall have to understand the difference.

The geometric Poisson distribution is a member of the general compound Poisson family. The geometric Poisson is a compound Poisson distribution where the compounding distribution $f(n)$ is given by the geometric distribution

$$f(n) = \begin{cases} (1-p)p^{n-1} & n \geq 1 \\ 0 & \text{otherwise} . \end{cases}$$

Note that each arrival must create at least one demand. The generating function $F(z)$ is obtained using Equation 3.4 as

$$F(z) = \sum_{n=0}^{\infty} f(n)\, z^n = \sum_{n=1}^{\infty} (1-p)\, p^{n-1}\, z^n = \frac{(1-p)z}{1-pz} \quad . \tag{5.29}$$

If we assume that the Poisson sampling process has the average arrival rate μ, then Equation 5.10 can be written as

$$H(z,t) = e^{-\mu t[1-F(z)]}$$

and for the compounding distribution of Equation 5.29 we find

$$H(z,t) = e^{-\mu t[1-\frac{(1-p)z}{1-pz}]} = e^{-\mu t[\frac{1-z}{1-pz}]} \quad . \tag{5.30}$$

The bivariate transform $P^b(z,s)$ of $p(n,t)$ is given by

$$P^b(z,s) = L[H(z,t)] \quad , \tag{5.31}$$

and so

$$P^b(z,s) = \frac{1}{s+\mu\frac{1-z}{1-pz}} = \frac{1-pz}{s+\mu - szp - \mu z} \tag{5.32}$$

Membership of the geometric Poisson distribution in the compound Poisson family makes it unnecessary to specify whether the time interval during which arrivals are counted is begun after an arrival or at random, in accordance with the following reasoning. Because the sampling distribution for a compound Poisson process is the simple Poisson process, it is invariant to counting beginning at random or after an event, as shown earlier. Since the probability of n demands in time t depends only on the sampling distribution and the time invariant compounding distribution, the counting interval for compound Poisson process may be begun after an event or at random with the same result.

The quantity $P^b(z,s)$ as given by Equation 5.32 again reduces to the bivariate transform for the simple Poisson distribution when $p=0$. Notice that the denominator of this expression is the same as that of Equation 5.26; however, the numerator is different. Let us compute the expected number of demands in time t for the geometric-Poisson using Equation 4.23 and $P^b(z,s)$ as given by Equation 5.32. Thus,

$$L[\bar{n}(t)] = \frac{\partial}{\partial z} P^b(z,s)\Big|_{z=1} = \frac{\partial}{\partial z}\left(\frac{1-pz}{s+\mu-szp-\mu z}\right)\Big|_{z=1} = \frac{\mu/(1-p)}{s^2} , \tag{5.33}$$

and

$$\bar{n}(t) = \frac{\mu}{1-p} \; t \tag{5.34}$$

The expected number of arrivals in time t for the distribution has the same linear growth term $\mu/(1-p) = \lambda$, as that for the stuttering Poisson given by Equation 5.28; however, it does not have the constant term $p/(1-p)$. Since we would not expect this constant to appear on physical grounds, we may ask if we can modify the stuttering Poisson to eliminate this term. Perhaps when this is done $P^b(z,s)$ and hence $p(n,t)$ will be the same for both distributions.

The difference between the distributions is apparently caused by the finite probability of purchase for the stuttering Poisson at $t=0$. If this were eliminated by requiring that the first arrival not be instantaneous, then it is possible that the two distributions would agree. The finite probability of arrivals in an interval of zero length is in fact caused by beginning the counting process after an arrival occurs. One way to eliminate this difficulty is to begin the process at a random time. If this is done then $p_R(n,t)$, the probability of n arrivals in a time t beginning at random, has a bivariate transform given by Equation 4.22 rather than by Equation 4.20. If we use $A(s)$ as defined by Equation 5.25, then

$$P_R^b(z,s) = \frac{1-pz}{s+\mu - szp - \mu z} \quad , \tag{5.35}$$

in complete agreement with $P^b(z,s)$ for the geometric Poisson as given by Equation 5.32.

Thus we have shown that if we are interested in demand generated during a time period beginning at random, the stuttering Poisson and the geometric Poisson are identical.

The Negative Binomial Distribution

Introduction

The negative binomial distribution has often been suggested as a model for demand processes. This section will show how several different probabilistic models will lead to a stochastic process governed by the negative binomial distribution.

Distribution of First Success in a Sequence of Bernoulli Trials

The first model we shall discuss is based upon a sequence of Bernoulli trials. Suppose that we have such a sequence with probability of success, p, and probability of failure, $q=1-p$. Let $f(k;r,p)$ be the probability that the r^{th} success will occur on trial $r+k$. Then $f(k;r,p)$ is also equal to the probability that k failures will take place in $r+k-1$ trials and a success will take place on the $(r+k)^{th}$ trial. It follows then that $f(k;r,p)$ is given by

$$f(k;r,p) = \binom{r+k-1}{k} p^{r-1} q^k \cdot p \quad , \qquad k \geq 0 \quad , \tag{5.36}$$

or

$$f(k;r,p) = \binom{r+k-1}{k} p^r q^k \quad , \qquad k \geq 0 \quad . \tag{5.37}$$

With some manipulation, Equation 5.37 can be placed in the form often seen for the negative binomial distribution. To see this let us investigate the combination of a objects taken b at a time. By definition, we have

$$\binom{a}{b} = \begin{cases} \dfrac{a!}{b!(a-b)!} = \dfrac{a(a-1)\dots(a-b+1)}{b!} & a \geq b \\ 0 & a < b \text{ or } b < 0 . \end{cases} \tag{5.38}$$

We may replace a by -a and extend the definition to obtain

$$\binom{-a}{b} = \frac{-a(-a-1)\dots(-a-b+1)}{b!}$$

$$= (-1)^b \frac{a(a+1)\dots(a+b-1)}{b!}$$

$$= (-1)^b \binom{a+b-1}{b} \quad , \tag{5.39}$$

or

$$\binom{a+b-1}{b} = (-1)^b \binom{-a}{b} \quad . \tag{5.40}$$

Using Equation 5.40 we may rewrite Equation 5.37 as

$$f(k;r,p) = (-1)^k \binom{-r}{k} p^r q^k \quad . \tag{5.41}$$

Finally, we obtain

$$f(k;r,p) = \binom{-r}{k} p^r (-q)^k \quad . \tag{5.42}$$

Equation 5.42 represents a commonly used form of the negative binomial distribution; it is the form responsible for the name of the distribution. We have seen that $f(k;r,p)$ may be interpreted as the probability that in a sequence of Bernoulli trials with probability of success, p, the r^{th} success will occur on the $(r+k)^{th}$ trial. An interesting special case is the one where r is set equal to 1; then, $f(k;1,p)$ is the probability that the first success will occur on trial $k+1$. This quantity is given by

$$f(k;1,p) = p\,q^k \quad , \qquad k \geq 0 \quad . \tag{5.43}$$

We recognize Equation 5.43 as the expression for the probability density function of a geometric distribution. Note that this geometric distribution has a non-zero probability for the value $k=0$, unlike the geometric distribution discussed above. Let us define $F(z;r,p)$, the z-transform of the negative binomial distribution, by

$$F(z;r,p) = \sum_{k=0}^{\infty} z^k f(k;r,p) \quad . \tag{5.44}$$

Then for $r=1$ the z-transform is given by

$$F(z;1,p) = \sum_{k=0}^{\infty} z^k pq^k = \frac{p}{1-qz} \quad . \tag{5.45}$$

Now that we have the z-transform of the geometric distribution, we can find its mean and variance. Its mean is given by the derivative of its z-transform at the point $z=1$. Let us denote the mean of the negative binomial by $\bar{k}(r,p)$, its second moment by $\overline{k^2}(r,p)$ and its variance by $\sigma_k^2(r,p)$. Then using Equation 5.45, we have

$$\bar{k}(1,p) = \frac{d}{dz} F(z;1,p) \Big|_{z=1} = q/p \quad , \tag{5.46}$$

$$\overline{k^2}(1,p) = \frac{d^2}{dz^2} F(z;1,p) \Big|_{z=1} + \bar{k}(1,p) = \frac{2q^2+qp}{p^2} \quad , \tag{5.47}$$

and

$$\sigma_k^2(1,p) = \overline{k^2}(1,p) - [\bar{k}(1,p)]^2 = q/p^2 \tag{5.48}$$

as the basic moments of the geometric distribution.

Let us now find the z-transform of the general negative binomial. Combining Equations 5.37 and 5.44, we obtain

$$F(z;r,p) = \sum_{k=0}^{\infty} z^k \binom{r+k-1}{k} p^r q^k \quad , \tag{5.49}$$

or

$$F(z;r,p) = p^r \sum_{k=0}^{\infty} \binom{r+k-1}{k} (qz)^k \quad . \tag{5.50}$$

To obtain $F(z;r,p)$ let us examine the following relations. From the binomial theorem we have

$$(a+b)^n = \sum_{j=0}^{\infty} \binom{n}{j} a^{n-j} b^j \quad . \tag{5.51}$$

Recall that $\binom{n}{j}$ as defined by Equation 5.38 is zero for $j>n$ and $j<0$. If n is replaced by $-n$ the binomial theorem is defined by

$$(a+b)^{-n} = \sum_{j=0}^{\infty} \binom{-n}{j} a^{-n-j} b^j \quad . \tag{5.52}$$

Equation 5.39 allows us to write Equation 5.52 in the form

$$(a+b)^{-n} = \sum_{j=0}^{\infty} \binom{n+j-1}{j} a^{-n-j}(-b)^{j} \quad . \tag{5.53}$$

If in Equation 5.53, we take a=1, b=-qz, n=r, and j=k, then the result is

$$\sum_{k=0}^{\infty} \binom{r+k-1}{k} (qz)^{k} = (1-qz)^{-r} \quad , \tag{5.54}$$

and thus F(z; r,p) may be expressed in closed form by

$$F(z;r,p) = \frac{p^r}{(1-qz)^r} \quad . \tag{5.55}$$

We have obtained in closed form the z-transform of the general negative binomial distribution.

Distribution of the Sum of Samples from a Geometric Distribution

It is informative to write Equation 5.55 in the form

$$F(z;r,p) = [F(z;1,p)]^{r} \tag{5.56}$$

by using Equation 5.45. We see that the z-transform of the negative bonomial distribution can be expressed as the r^{th} power of the z-transform of the geometric distribution. It follows that the negative binomial distribution with parameter r may be interpreted as the distribution of the sum of r independent samples from the geometric distribution expressed by Equation 5.43.

The moments of the negative binomial distribution are calculated from its z-transform as follows:

$$\bar{k}(r,p) = \frac{d}{dz} F(z;r,p) \Big|_{z=1} = r(q/p) \quad , \tag{5.57}$$

$$\overline{k^2}(r,p) = \frac{d^2}{dz^2} F(z;r,p) \Big|_{z=1} + \bar{k}(r,p) = \frac{r^2q^2 + rq^2 + rpq}{p^2} \quad , \tag{5.58}$$

and

$$\sigma_k^2(r,p) = \overline{k^2}(r,p) - [\bar{k}(r,p)]^2 = r(q/p^2) \quad . \tag{5.59}$$

We see that these results not only agree with those of Equations 5.46, 5.47, and 5.48 when $r=1$, but also that they support our contention that the negative binomial distribution may be interpreted as the sum of independent samples from the same geometric distribution. Thus, the mean of the negative binomial is r times the mean of the underlying geometric distribution, while the variance of the negative binomial is r times the variance of the geometric.

We have now obtained two completely equivalent expressions for the negative binomial distribution given by $f(k;r,p)$. First, it represents the probability that in a sequence of Bernoulli trials with probability of success p, the r^{th} success will occur on the $(r+k)^{th}$ trial. Second, it represents the probability that the sum of r independent samples from a geometric distribution with parameter p will take on the value k. Let us now proceed to another process that gives rise to a negative binomial distribution.

Poisson Demands with Erlang Arrival Rates

Suppose that the demand for a product in a given time interval of length t is Poisson-distributed with mean rate λ. Then the probability of n demands in the period, $d(n|\lambda t)$, is given by

$$d(n|\lambda t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \quad . \tag{5.60}$$

Suppose further that the demand rate, λ, in successive periods of length t is determined by independent samples from the density function $g(\lambda)$ defined by

$$g(\lambda) = \frac{(k\mu)^k \lambda^{k-1} e^{-k\lambda\mu}}{(k-1)!} \quad , \qquad 0<\lambda<\infty \quad . \tag{5.61}$$

That is, the demand rates from period to period are chosen from a k phase Erlang distribution with mean $\frac{1}{\mu}$. Of course, as k becomes infinite $g(\lambda)$ tends to a unit impulse at $\frac{1}{\mu}$, and the demand process will be a simple Poisson with mean rate $\lambda = \frac{1}{\mu}$ in each period. Let $d(n,t)$ be the probability that n units are demanded in a given period of length t. Then $d(n,t)$ is given by

$$d(n,t) = \int_0^\infty d(n|\lambda t)\, g(\lambda)\, d\lambda = \int_0^\infty \frac{(\lambda t)^n e^{-\lambda t}}{n!} \; \frac{(k\mu)^k \lambda^{k-1} e^{-k\lambda\mu}}{(k-1)!}\, d\lambda, \tag{5.62}$$

or

$$d(n,t) = \frac{(k\mu)^k}{n!(k-1)!\,t^k} \int_0^\infty (\lambda t)^{n+k-1} e^{-\left(\frac{t+k\mu}{t}\right)\lambda t}\, d(\lambda t) \quad . \tag{5.63}$$

Using the gamma function integral expressed by

$$\int_0^\infty x^m e^{-ax}\, dx = \frac{\Gamma(m+1)}{a^{m+1}} = \frac{m!}{a^{m+1}} \quad , \tag{5.64}$$

we obtain

$$d(n,t) = \frac{(k\mu)^k}{n!(k-1)!\,t^k} \; \frac{(n+k-1)!\,t^{n+k}}{(t+k\mu)^{n+k}}$$

$$= \binom{n+k-1}{n} \left(\frac{t}{t+k\mu}\right)^n \left(\frac{k\mu}{t+k\mu}\right)^k \quad . \tag{5.65}$$

Comparison of Equation 5.65 with Equation 5.37 shows that the number of demands in a time interval of length t is given by a negative binomial distribution. The parameter r of the negative binomial is the number of phases, k, in the Erlang distribution; the probability of success, p, for the negative binomial is given by $\frac{k\mu}{t+k\mu}$. Thus we have

$$d(n,t) = f(n; k, \frac{k\mu}{t+k\mu}) \quad . \tag{5.66}$$

We have thus shown that if demand in successive time intervals is Poisson with mean rates selected as independent samples from the Erlang distribution, then the number of demands in a particular interval will be distributed according to the negative binomial distribution. Such a distribution could arise in the Navy Supply System, for example, if the number of ships in port at any particular time followed the Erlang distribution, and each ship generated demands according to the same Poisson distribution. The total quantity demanded of the Navy Supply System would then be distributed according to the negative binomial distribution.

The Negative Binomial as a Compound Poisson

Still another model of the negative binomial is based on the compound Poisson distribution. Suppose that arrivals are governed by a Poisson process with mean rate λ and that when each arrival occurs demand is generated as a sample from a discrete distribution $f(n)$ with z-transform $F(z)$. From the theory of the compound Poisson distribution developed above, we have from Equation 5.10 that the z-transform of the number of demands in time t is

$$H(z,t) = e^{-\lambda t[1-F(z)]} \tag{5.67}$$

If the compound Poisson demand is to be governed by a negative binomial distribution with parameters r and p, then its z-transform must also be given by Equation 5.55. Hence, if we set Equation 5.55 equal to Equation 5.67 we may solve for $F(z)$, the z-transform of the required compounding distribution. Thus, we write

$$e^{-\lambda t[1-F(z)]} = \left(\frac{p}{1-qz}\right)^r , \tag{5.68}$$

and solve for $F(z)$. Inversion of the z-transform $F(z)$ will then yield the analytic expression for the compounding distribution required in the compound Poisson so that total demand will be given by the negative binomial distribution. Proceeding algebraically, we obtain

$$-\lambda t[1-F(z)] = r\ln p - r\ln(1-qz)$$

$$1-F(z) = -\frac{r}{\lambda t}\ln p + \frac{r}{\lambda t}\ln(1-qz)$$

$$F(z) = 1 + \frac{r}{\lambda t}\ln p - \frac{r}{\lambda t}\ln(1-qz) \quad . \tag{5.69}$$

Using the logarithmic expansion

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} , \qquad |x| < 1 \tag{5.70}$$

we may rewrite Equation 5.69 as

$$F(z) = 1 + \frac{r}{\lambda t}\ln p + \frac{r}{\lambda t}\sum_{n=1}^{\infty}\frac{(qz)^n}{n} \quad . \tag{5.71}$$

The z-transform of the compounding distribution is in a particularly convenient form as it is given by Equation 5.71. Since $F(z)$ is related to the compounding distribution $f(n)$ by

$$F(z) = \sum_{n=0}^{\infty} z^n f(n) \quad , \tag{5.72}$$

identification of the terms of Equations 5.71 and 5.72 allows to write an explicit form for the compounding distribution as follows:

$$f(0) = 1 + \frac{r}{\lambda t} \ln p$$

$$f(n) = \frac{r}{\lambda t} \frac{q^n}{n} \quad , \qquad n \geq 1 . \tag{5.73}$$

Thus, a compound Poisson distribution will produce a demand governed by the negative binomial distribution if the compounding distribution is of the so-called "logarithmic" form. We see from Equations 5.73 that there is a finite probability, $1 + \frac{r}{\lambda t} \ln p$, that no demands will be made when an arrival occurs. If we would like to require at least one demand from each arrival, then $f(0)$ must be made equal to 0. This can be accomplished by setting

$$p = e^{-\lambda t/r} \quad . \tag{5.74}$$

Of course, it is required that $f(0)$ never be less than 0 or greater than 1. Thus, we may write

$$0 \leq 1 + \frac{r}{\lambda t} \ln p \leq 1 \quad , \tag{5.75}$$

which may be written as

$$e^{-\lambda t/r} \leq p \leq 1 \quad . \tag{5.76}$$

Suppose now that one wished to represent a negative binomial process with parameters r and p by means of a compound Poisson process with Poisson rate λ . Equations 5.73 show how the compounding distribution should be chosen as a function of the observation interval, t , and the given parameters. However, it will not always be possible to find a compounding distribution for arbitrary r, p, λ, and t because f(0) may be negative. To insure that this is not the case, r, p, λ, and t must meet the requirements imposed by Equation 5.76. Since for given r, p, and λ, Equation 5.76 can always be satisfied by taking t sufficiently large, we can say that an arbitrary negative binomial distribution can be represented by a compound Poisson distribution with an arbitrary Poisson rate.

Let us find the moments of the compounding distribution given by Equations 5.73. Its z-transform may be written from Equation 5.69 as

$$F(z) = 1 + \ln \left[\frac{p}{1-qz}\right]^{r/\lambda t} \qquad (5.77)$$

If we refer to the mean, second moment, and variance of the compounding distribution by $\bar{n}$, $\overline{n^2}$, and σ_n^2 , we obtain

$$\bar{n} = \left.\frac{dF(z)}{dz}\right|_{z=1} = \frac{r}{\lambda t}\, q/p$$

$$\overline{n^2} = \left.\frac{d^2F(z)}{dz^2}\right|_{z=1} + \bar{n} = \frac{r}{\lambda t}\, q/p^2 \qquad (5.78)$$

$$\sigma_n^2 = \overline{n^2} - \bar{n}^2 = \frac{rq}{\lambda t p^2}\left[1 - \frac{rq}{\lambda t}\right]$$

The mean, $\bar{m}$, and variance, σ_m^2, of the compound Poisson distribution are determined by the moments of the compounding distribution as shown in

Equations 5.16 and 5.17. We find

$$\bar{m} = \lambda t\, \bar{n} = \lambda t \frac{rq}{\lambda tp} = rq/p \quad , \tag{5.79}$$

and

$$\sigma_m^2 = \lambda t\, \overline{n^2} = \lambda t \frac{r}{\lambda t} \frac{q}{p^2} = \frac{rq}{p^2} \tag{5.80}$$

Comparison of Equations 5.57 and 5.59 with Equations 5.79 and 5.80 show that the compound Poisson distribution constructed according to Equation 5.73 has the same mean and variance as the original negative binomial distribution. Indeed, the two distributions will be identical in every respect.

Summary

We have now shown four possible ways in which the negative binomial distribution could be generated.

1. As the probability of obtaining the r^{th} success on the $(r+k)^{th}$ Bernoulli trial.
2. As the probability distribution of the sum of r independent samples from the geometric distribution.
3. As the distribution of total demand where demand in a particular time interval is Poisson but where the Poisson demand rate is chosen from interval according to an Erlang distribution.
4. As a compound Poisson distribution where the compounding distribution is logarithmic in form.

Other models are no doubt possible; however, it would be worthwhile in any inventory situation in which a negative binomial demand is experienced to see if one of these four models can explain the underlying phenomenon.

This chapter has described the general compound Poisson distribution and important members of the family of such distributions. We have seen two examples in which different mathematical models have led to the same observed demand process. The compound Poisson family is particularly important because it is the only family of discrete distributions that satisfies the assumption of independent demands in arbitrary independent time periods. That is, it is the only discrete distribution that satisfies

$$p(n,t) = \sum_{m=0}^{n} p(m,\tau)\, p(n-m,\, t-\tau) \quad , \qquad 0 < \tau < t \ .^{1/} \tag{5.81}$$

This assumption is frequently made in inventory theory, often without the realization that it implies the compound Poisson demand process.

1/ See Feller, W., An Introduction to Probability Theory and Its Applications Volume I, Second Edition, Wiley 1958, pp. 270 - 272.

VI. THE ERLANG DISTRIBUTION

Introduction

An important recurrent process is that in which the density function for the time between events, $a(\tau)$, is the Erlang distribution defined by Equation 6.1,

$$a(\tau) = \frac{k\lambda(k\lambda\tau)^{k-1}e^{-k\lambda\tau}}{(k-1)!} \quad , \qquad \tau \geq 0 \ . \qquad (6.1)$$

The Erlang density function has two parameters k and λ. The quantity k may be integral or non-integral; when it is non-integral the factorial in Equation 6.1 must be replaced by the corresponding gamma function. Without loss of generality we shall consider k to be integral in which case the Erlang distribution can be considered to be the density function of the sum of k independent samples from an exponential distribution with mean $\frac{1}{k\lambda}$. It then follows that the mean of the Erlang distribution must be given by $\frac{1}{\lambda}$, and its variance by $\frac{1}{k\lambda^2}$. If we compare Equation 6.1 with the Poisson density function, Equation 5.4, and the Laplace transform of the Poisson density function, Equation 5.3, the Laplace transform of the Erlang distribution can be written by inspection as

$$A(s) = \left[\frac{k\lambda}{s+k\lambda}\right]^k \qquad (6.2)$$

If we substitute Equation 6.2 into Equation 4.7 we obtain the Laplace transform of the probability that n events will occur in a time interval t after an event as

$$P(n,s) = \frac{1}{s}\left[1 - \left(\frac{k\lambda}{s+k\lambda}\right)^k\right]\left(\frac{k\lambda}{s+k\lambda}\right)^{nk} \quad , \qquad n \geq 0 \ . \qquad (6.3)$$

We must now invert this transform to obtain an explicit expression for $p(n,t)$.

To perform this inversion, let us consider the truncated exponential function, $E_n(x)$, defined by

$$E_n(x) = e^{-x} \sum_{j=0}^{n} \frac{x^j}{j!} \quad , \qquad n \geq 0 \ . \tag{6.4}$$

This function is discussed and tabulated by Morse[1/]. We shall now find the Laplace transform of $E_n(x)$. We first write

$$e^x E_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!} \quad , \qquad n \geq 0 \ . \tag{6.5}$$

and then

$$L[e^x E_n(x)] = \sum_{j=0}^{n} L[\frac{x^j}{j!}] = \frac{1}{s} \sum_{j=0}^{n} \frac{1}{s^j}$$

$$= \frac{1}{s} \frac{1-(1/s)^{n+1}}{1-1/s}$$

$$= \frac{1}{s-1} [1 - \frac{1}{s^{n+1}}] \quad , \qquad n \geq 0 \ . \tag{6.6}$$

However,

$$L[E_n(x)] = L[e^x E_n(x)]\Big|_{s = s+1} \quad , \tag{6.7}$$

1/ P.M. Morse, Queues, Inventories and Maintenance, Wiley, New York, 1958.

and so

$$L[E_n(x)] = \frac{1}{s}\left[1 - \frac{1}{(s+1)^{n+1}}\right], \qquad n \geq 0 . \tag{6.8}$$

We have thus obtained the Laplace transform of the truncated exponential function defined by Equation 6.4. If the truncated exponential function should be written in the form

$$E_n(ax) = e^{-ax} \sum_{j=0}^{n} \frac{(ax)^j}{j!} , \tag{6.9}$$

where a is a constant, then we can find the corresponding transform by making use of the Laplace transform property

$$L[E_n(ax)] = \frac{1}{a} L[E_n(x)]\Big|_{s=s/a} . \tag{6.10}$$

By using the results of Equations 6.8 and 6.10 we can finally write

$$L[E_n(ax)] = \frac{1}{s}\left[1 - \frac{a^{n+1}}{(s+a)^{n+1}}\right], \qquad n \geq 0 . \tag{6.11}$$

We can now proceed to the inversion of the transform expression in Equation 6.3. From Equation 6.3 with $n = 0$ we obtain the transform of $p(0,t)$, the probability that no events will occur in time t, as

$$P(0,s) = \frac{1}{s}\left[1 - \left(\frac{k\lambda}{s+k\lambda}\right)^k\right] . \tag{6.12}$$

Comparison of Equations 6.11 and 6.12 then shows that $p(0,t)$ must be given by

$$p(0,t) = E_{k-1}(k\lambda t) = e^{-k\lambda t} \sum_{j=1}^{k-1} \frac{(k\lambda t)^j}{j!} . \tag{6.13}$$

When n is larger than 0, we write Equation 6.3 in the form

$$
\begin{aligned}
P(n,s) &= \frac{1}{s}\left[1-\left(\frac{k\lambda}{s+k\lambda}\right)^{k}\right]\left(\frac{k\lambda}{s+k\lambda}\right)^{nk} \\
&= \frac{1}{s}\left[\left(\frac{k\lambda}{s+k\lambda}\right)^{nk}-\left(\frac{k\lambda}{s+k\lambda}\right)^{(n+1)k}\right] \\
&= \frac{1}{s}\left[1-\left(\frac{k\lambda}{s+k\lambda}\right)^{(n+1)k}\right]-\frac{1}{s}\left[1-\left(\frac{k\lambda}{s+k\lambda}\right)^{nk}\right] \quad . \qquad (6.14)
\end{aligned}
$$

If we use Equation 6.11 we can write $p(n,t)$ the inverse transform of $P(n,s)$ as

$$p(n,t) = E_{(n+1)k-1}(k\lambda t) - E_{nk-1}(k\lambda t) \quad , \qquad n \geq 1 \quad . \qquad (6.15)$$

Equation 6.15 may be written in the form

$$p(n,t) = e^{-k\lambda t}\left[\sum_{j=0}^{[(n+1)k-1]} \frac{(k\lambda t)^j}{j!} - \sum_{j=0}^{nk-1} \frac{(k\lambda t)^j}{j!}\right] \quad , \qquad (6.16)$$

and finally, in the form

$$p(n,t) = e^{-k\lambda t} \sum_{j=nk}^{[(n+1)k-1]} \frac{(k\lambda t)^j}{j!} \quad , \qquad n \geq 0 \quad . \qquad (6.17)$$

The expression for $p(n,t)$ given by Equation 6.17 is valid for all n greater than or equal to 0.

We have thus obtained an explicit expression for the probability that n events will occur in a time interval t following a given event for a recurrent event process with an Erlang interevent time density function. We may provide an important interpretation for this expression when we note that the truncated exponential function defined by Equation 6.4 is equal to the probability that a sample from a Poisson distribution of mean x will be less than or equal to n. To use this fact, we shall build the following model of the process under consideration. Let us imagine a sequence of recurrent events which we shall call blips. The interblip time density function is exponential with mean $\frac{1}{k\lambda}$; consequently, blips are generated by a Poisson process of rate $k\lambda$. If we say that an "event" occurs at every k^{th} blip then the events will be a recurrent process with the Erlang interevent time density function of Equation 6.1. The probability that no event will occur in a time interval t following an event is the probability that the number of blips in the time interval t will be less than or equal to k-1. Since the mean number of blips in time t is $k\lambda t$, Equation 6.13 is verified immediately.

The probability that there will be exactly $n \geq 1$ events in a time interval t, following a given event, is the probability that the number of blips in this time interval will be greater than or equal to nk, but less than or equal to $(n+1)k-1$. It is thus equal to the probability that there will be $(n+1)k-1$ or fewer blips less the probability that there will be $nk-1$ or fewer blips. This relation is expressed exactly by Equation 6.15.

Now that we have verified our results, let us obtain some other properties of the Erlang process. Let $P(r,t)$ be the probability that r or fewer events will occur in the time interval t. Thus $P(r,t)$ is defined by

$$P(r,t) = \sum_{n=0}^{r} p(n,t) \quad . \tag{6.18}$$

Equation 6.17 allows us to write

$$P(r,t) = \sum_{n=0}^{r} e^{-k\lambda t} \sum_{j=nk}^{[(n+1)k-1]} \frac{(k\lambda t)^j}{j!}$$

$$= e^{-k\lambda t} \sum_{n=0}^{r} \sum_{j=nk}^{[(n+1)k-1]} \frac{(k\lambda t)^j}{j!}$$

$$= e^{-k\lambda t} \sum_{j=0}^{[(r+1)k-1]} \frac{(k\lambda t)^j}{j!}$$

$$= E_{(r+1)k-1}(k\lambda t) \quad , \qquad r \geq 0 \quad . \qquad (6.19)$$

The cumulative probability of events is thus also given by a truncated exponential function. Once more, the physical interpretation is clear. The probability that there will be r or fewer events in time t is just the probability that there will be $(r+1)k-1$ or fewer blips in time t.

An expression for the expected number of events in time t, $\bar{n}(t)$, is also easily obtained. This quantity is defined by

$$\bar{n}(t) = \sum_{n=1}^{\infty} np(n,t) \quad . \qquad (6.20)$$

Equation 6.20 can be written in the form

$$\bar{n}(t) = \sum_{n=1}^{\infty} p(n,t) \sum_{m=1}^{n} 1 = \sum_{n=1}^{\infty} \sum_{m=1}^{n} p(n,t) \quad . \qquad (6.21)$$

By reversing the order of summation we obtain

$$\bar{n}(t) = \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} p(n,t) = \sum_{m=1}^{\infty} \left[1 - \sum_{n=0}^{m-1} p(n,t) \right]$$

$$= \sum_{m=1}^{\infty} [1 - P(m-1,t)]$$

$$= \sum_{m=1}^{\infty} [1 - E_{mk-1}(k\lambda t)] \quad . \tag{6.22}$$

Thus, an expression for the expected number of events in time t is obtained not in closed form but as an infinite summation of terms that are tabulated as the probability that a sample from the Poisson distribution will exceed a specified value. Recall that if sampling were begun at random, the expected number of events in time t would be simply λt, as expressed in Equation 4.27.

The second moment of the number of events in time t after an event, $\overline{n^2}(t)$ is calculated in a similar fashion. The defining relation is

$$\overline{n^2}(t) = \sum_{n=1}^{\infty} n^2 \, p(n,t) \quad . \tag{6.23}$$

Proceeding as before, we write

$$\overline{n^2}(t) = \sum_{n=1}^{\infty} p(n,t) \sum_{m=1}^{n} (2m-1) = \sum_{n=1}^{\infty} \sum_{m=1}^{n} (2m-1) \, p(n,t) \quad . \tag{6.24}$$

The same change in order of summation yields,

$$\overline{n^2}(t) = \sum_{m=1}^{\infty} (2m-1) \sum_{n=m}^{\infty} p(n,t)$$

$$= \sum_{m=1}^{\infty} (2m-1) \left[1 - \sum_{n=0}^{m-1} p(n,t) \right]$$

$$= \sum_{m=1}^{\infty} (2m-1) \; [1 - P(m-1,t)]$$

$$= \sum_{m=1}^{\infty} (2m-1) \; [1 - E_{mk-1} (k\lambda t)] \quad . \tag{6.25}$$

The expression for the second moment of the number of events in time t after an event involves summation over the same tabulated terms required for the calculation for the first moment; however, these terms are now weighted by an additional factor.

An interesting compound process results when the sampling process is the recurrent Erlang that we have been discussing. In the compound process at each event time some number of hyperevents determined by a compounding distribution also occurs. For an arbitrary compounding distribution it is quite difficult to determine the probability that a given number of hyperevents will occur in a time interval t. However, the mean and variance of the number that will occur can be calculated by knowing the first and second moments of the compounding distribution and the first and second moments of the sampling distribution. These latter quantities are given for the compound Erlang distribution by Equations 6.21 and 6.25 respectively. Consequently, with some numerical effort the important moments of a compound Erlang distribution can be obtained.

APPENDIX

DEVELOPMENT OF THE COMPOUND DISTRIBUTION FOR A COMPOUNDING DISTRIBUTION HAVING FOR ITS DOMAIN ANY REAL NUMBER

This appendix will develop the theory of the compound distribution for a continuous compounding distribution $f(n)$. As in Chapter III, let m be the sum of k samples from the density function $f(n)$. We wish to find the compound distribution $h(m)$ generated by the functions $g(k)$, the sampling distribution, and $f(n)$. From Equation 3.1 we have

$$h(m) = \sum_{k=0}^{\infty} g(k)\, h(m|k) \quad . \tag{A.1}$$

Since m is the sum of k independent samples of $f(n)$, the probability density $h(m|k)$ is given by the k-fold convolution of $f(n)$, denoted, as before, by $f^{k*}(m)$. We define

$$f^{k*}(m) = \int_{-\infty}^{\infty} f(x)\, f^{(k-1)*}(m-x)\, dx \quad , \qquad k \geq 1 \quad ,$$

and

$$f^{0*}(m) = \begin{cases} 1 & \text{for } m=0 \\ 0 & \text{otherwise} \end{cases} \quad . \tag{A.2}$$

We can now write Equation A.1 as

$$h(m) = \sum_{k=0}^{\infty} g(k)\, f^{k*}(m) \quad . \tag{A.3}$$

Let $H(j\omega)$ and $F(j\omega)$ ($j=\sqrt{-1}$) be the Fourier transforms of the density functions $h(m)$ and $f(n)$, defined by

$$H(j\omega) = \int_{-\infty}^{\infty} h(m)\, e^{-j\omega m}\, dm \quad ,$$

and

$$F(j\omega) = \int_{-\infty}^{\infty} f(n)\, e^{-j\omega n}\, dn \quad . \qquad (A.4)$$

Also, let us define the z-transform of the sampling distribution, $G(z)$ as

$$G(z) = \sum_{k=0}^{\infty} g(k)\, z^k \quad . \qquad (A.5)$$

If we multiply both sides of Equation A.3 by $e^{-j\omega m}$ and integrate over all m, we obtain

$$\int_{-\infty}^{\infty} h(m)\, e^{-j\omega m}\, dm = \sum_{k=0}^{\infty} g(k) \int_{-\infty}^{\infty} f^{k*}(m)\, e^{-j\omega m}\, dm \quad . \qquad (A.6)$$

Since the Fourier transform of the k-fold convolution of $f(n)$ is the Fourier transform of $f(n)$ raised to the k^{th} power, then using the definitions of Equation A.4, we may write Equation A.6 as

$$H(j\omega) = \sum_{k=0}^{\infty} g(k)\, [F(j\omega)]^k \quad . \qquad (A.7)$$

The definition of $G(z)$ in Equation A.5 allows us to write

$$H(j\omega) = G\,[F(j\omega)] \quad . \qquad (A.8)$$

This shows that the Fourier transform of the compound distribution, $h(m)$, is the z-transform of the sampling distribution, taking z to be the Fourier transform of the compounding distribution.

Let us find the moments of the compound distribution in terms of the moments of the sampling and compounding distributions. The r^{th} moment of a distribution is given in terms of its Fourier transform, $B(j\omega)$, as

$$\overline{x^r} = j^r \frac{\partial^r}{\partial \omega^r} B(j\omega) \Big|_{\omega=0} \quad . \tag{A.9}$$

Then, using Equation A.9, we find

$$\bar{n} = j \frac{\partial}{\partial \omega} F(j\omega) \Big|_{\omega=0} = jF'_\omega(0) \quad , \tag{A.10}$$

$$\overline{n^2} = j^2 \frac{\partial^2}{\partial \omega^2} F(j\omega) \Big|_{\omega=0} = j^2 F''_\omega(0) \quad , \tag{A.11}$$

and

$$\sigma_n^2 = j^2 F''_\omega(0) - [jF'_\omega(0)]^2 \quad . \tag{A.12}$$

The mean, $\bar{m}$, and variance, σ_m^2, of the compound distribution, $h(m)$, may be obtained as follows:

$$H(j\omega) = G[F(j\omega)] \tag{A.13}$$

$$jH'_\omega(j\omega) = G'[F(j\omega)] \; jF'_\omega(j\omega) \tag{A.14}$$

$$\bar{m} = jH'_\omega(j\omega) \Big|_{\omega=0} = G'[F(0)] \; jF'_\omega(0) \quad . \tag{A.15}$$

However,

$$F(0) = \int_{-\infty}^{\infty} f(n)\, dn = 1 \quad . \tag{A.16}$$

Therefore,

$$\begin{aligned} \bar{m} &= G'(1)\, \bar{n} \\ &= \bar{k}\, \bar{n} \quad . \end{aligned} \tag{A.17}$$

This result agrees with Equation 3.11 for the case of discrete compounding distributions.

For the variance we compute

$$\sigma_m^2 = \overline{m^2} - \bar{m}^2 = \left\{ j^2 H_\omega''(j\omega) - [jH_\omega'(j\omega)]^2 \right\} \Bigg|_{\omega=0} \quad . \tag{A.18}$$

First, let us evaluate

$$j^2 H_\omega''(j\omega) = G''\,[F(j\omega)]\,[jF_\omega'(j\omega)]^2 + G'[F(j\omega)]\; j^2 F_\omega''(j\omega) \quad , \tag{A.19}$$

or

$$\sigma_m^2 = G''\,[F(0)]\,\bar{n}^2 + G'[F(0)]\,\overline{n^2} - (\bar{k}\,\bar{n})^2 \quad . \tag{A.20}$$

But

$$G''(1) = \sigma_k^2 - \bar{k} + \bar{k}^2 \quad . \tag{A.21}$$

Therefore,

$$\sigma_m^2 = [\sigma_k^2 - \bar{k} + \bar{k}^2]\, \bar{n}^2 + \bar{k}\, \overline{n^2} - \bar{k}^2\, \bar{n}^2$$

$$= \sigma_k^2 \bar{n}^2 - \bar{k}\,\bar{n}^2 + \bar{k}^2\,\bar{n}^2 + \bar{k}\,\overline{n^2} - \bar{k}^2\,\bar{n}^2$$

$$= \sigma_k^2 \bar{n}^2 + (\bar{k}\,\overline{n^2} - \bar{k}\,\bar{n}^2)$$

$$= \bar{n}^2\,\sigma_k^2 + \bar{k}\,\sigma_n^2 \qquad \text{(A.22)}$$

Again, this result agrees exactly with Equation 3.13 for the corresponding discrete case.